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Exact solutions for the generalized nonlinear heat conduction equations using the exp-function method

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ABSTRACT

The exp-function method is used to find exact solutions of the generalized nonlinear heat conduction equations. The results obtained include all the solutions from the open literature as special cases.

1. Introduction

In this paper, we consider the following nonlinear heat conduction equation [1]:

$$u_t - a\left(u^n\right)_{xx} - u + u^n = 0, \quad a > 0,\ n > 1, \tag{1}$$

where a is a constant.

The equation was studied by Wazwaz using the tanh method [1]. Alternative approaches to the problem include the homotopy perturbation method [2], the variational method [3,4], and the variational iteration method [5,6]. In this paper, the exp-function method [7] is applied to search for generalized solitary solutions of Eq. (1).

2. Exp-function method

The exp-function method [7] has become one of the best candidates for finding exact solutions of nonlinear equations [8–15].

Introducing a complex variation defined as

$$u = u\left(\eta\right), \quad \eta = kx + \omega t, \tag{2}$$

where k and ω are constants, and substituting Eq. (2) into Eq. (1), we have

$$\omega u' - ak^2\left(u^n\right)'' - u + u^n = 0, \quad a > 0,\ n > 1. \tag{3}$$

Using the transformation $u = v^{-\frac{1}{n-1}}$, we can rewrite Eq. (3) as the following equation:

$$\frac{\omega(n-1)}{3}\left(v^3\right)' + (n-1)^2 v^2\left(v-1\right) + ak^2 n\left(2n-1\right)\left(v'\right)^2 - ak^2 n(n-1) v v'' = 0. \tag{4}$$

According to the exp-function method, we assume that the solution can be expressed in the following form [7]:

$$v\left(\eta\right) = \frac{a_1 \exp\left(\eta\right) + a_0 + a_{-1}\exp\left(-\eta\right)}{\exp\left(\eta\right) + b_0 + b_{-1}\exp\left(-\eta\right)}. \tag{5}$$

Substituting Eq. (5) into Eq. (4), and with the help of a symbolic computation system, we have

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$$\frac{1}{A}\{C_0 + C_1\exp(\eta) + C_2\exp(2\eta) + \cdots + C_7\exp(7\eta) + C_8\exp(8\eta)\} = 0. \tag{6}$$

Equating the coefficients of $\exp(n\eta)$ in Eq. (6) to zero yields a set of algebraic equations as follows:

$$C_0 = 0,\quad C_1 = 0,\quad C_2 = 0,\quad C_3 = 0,\quad C_4 = 0,\quad C_5 = 0,\quad C_6 = 0,\quad C_7 = 0,\quad C_8 = 0. \tag{7}$$

Solving the system, Eq. (7), simultaneously leads to

$$\text{Case 1: } a_1 = 0, a_0 = a_0, a_{-1} = a_{-1}, b_0 = \frac{a_0^2 + a_{-1}}{a_0}, b_{-1} = a_{-1}, k = \pm\frac{n-1}{n\sqrt{a}}, \omega = \frac{1-n}{n}, \tag{8}$$

$$\text{Case 2: } a_1 = 0, a_0 = b_0, a_{-1} = b_0^2, b_0 = b_0, b_{-1} = 0, k = \pm\frac{n-1}{n\sqrt{a}}, \omega = n-1, \tag{9}$$

where the constant a can't be zero.

We therefore obtain two solitary solutions, which read

$$u_1(x,t) = \left(\frac{a_0}{\exp\left[\pm\frac{n-1}{n\sqrt{a}}x + \frac{1-n}{n}t\right] + a_0}\right)^{\frac{-1}{n-1}} \tag{10}$$

and

$$u_2(x,t) = \left(\frac{b_0}{\exp\left[\pm\frac{n-1}{n\sqrt{a}}x + (n-1)t\right]}\right)^{\frac{-1}{n-1}}. \tag{11}$$

To compare our results with those obtained in [1], we set the parameters in Eq. (10), respectively, as follows:

$$\begin{aligned}
&(1)\quad a_0 = 1,\quad k = \frac{n-1}{\sqrt{a}n};\\
&(2)\quad a_0 = -1,\quad k = \frac{n-1}{\sqrt{a}n};\\
&(3)\quad a_0 = 1,\quad k = -\frac{n-1}{\sqrt{a}n};\\
&(4)\quad a_0 = -1,\quad k = -\frac{n-1}{\sqrt{a}n}.
\end{aligned} \tag{12}$$

We therefore obtain

$$u_{1(1)}(x,t) = \left(1 + \exp\left[\frac{(n-1)}{n\sqrt{a}}\left(x - \sqrt{a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{n-1}{2n\sqrt{a}}\left(x - \sqrt{a}t\right)\right]\right)^{\frac{-1}{n-1}} \tag{13}$$

$$u_{1(2)}(x,t) = \left(1 - \exp\left[\frac{(n-1)}{n\sqrt{a}}\left(x - \sqrt{a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} - \frac{1}{2}\coth\left[\frac{n-1}{2n\sqrt{a}}\left(x - \sqrt{a}t\right)\right]\right)^{\frac{-1}{n-1}} \tag{14}$$

$$u_{1(3)}(x,t) = \left(1 + \exp\left[\frac{(1-n)}{n\sqrt{a}}\left(x + \sqrt{a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{n-1}{2n\sqrt{a}}\left(x + \sqrt{a}t\right)\right]\right)^{\frac{-1}{n-1}} \tag{15}$$

$$u_{1(4)}(x,t) = \left(1 - \exp\left[\frac{(1-n)}{n\sqrt{a}}\left(x + \sqrt{a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} + \frac{1}{2}\coth\left[\frac{n-1}{2n\sqrt{a}}\left(x + \sqrt{a}t\right)\right]\right)^{\frac{-1}{n-1}}. \tag{16}$$

These are the solutions obtained by the tanh method in [1].

Setting $n = 3$, we obtain the solutions of the standard form of the nonlinear heat conduction equation as follows:

$$u_1(x,t) = \left(\frac{a_0}{\exp\left[\pm\frac{1}{2\sqrt{a}}x - \frac{1}{2}t\right] + a_0}\right)^{-\frac{1}{2}}, \tag{17}$$

$$u_2(x,t) = \left(\frac{b_0}{\exp\left[\pm\frac{1}{2\sqrt{a}}x - \frac{1}{2}t\right]}\right)^{-\frac{1}{2}}. \tag{18}$$

The solution, Eq. (17), includes four solutions from Ref. [1] on suitably setting the free parameters, while Eq. (18) is new.

3. The nonlinear generalized heat conduction equation in two dimensions

In this section, we consider a generalized form of the nonlinear heat conduction equation in $(2+1)$-dimensional space, which reads [1]

$$u_t - a\left(u^n\right)_{xx} - a\left(u^n\right)_{yy} - u + u^n = 0, \quad a > 0,\ n > 1. \tag{19}$$

Introducing a complex variation defined as

$$u = u(\eta), \quad \eta = kx + \omega t, \tag{20}$$

where k and ω are constants, Eq. (19) becomes

$$\omega u' - a\left(k^2+1\right)\left(u^n\right)'' - u + u^n = 0, \quad a > 0,\ n > 1. \tag{21}$$

Using the transformation $u = v^{-\frac{1}{n-1}}$, we have

$$\frac{\omega(n-1)}{3}\left(v^3\right)' + (n-1)^2 v^2 (v-1) + a\left(k^2+1\right)\left(n(2n-1)\left(v'\right)^2 - n(n-1)vv''\right) = 0. \tag{22}$$

By the same manipulation as is illustrated in Section 2, we obtain the solitary wave solutions as follows:

$$u_3(x,y,t) = \left(\frac{a_0}{\exp[kx+y+\omega t] + a_0}\right)^{\frac{-1}{n-1}}, \quad k = \pm\sqrt{\frac{(n-1)^2}{an^2} - 1},\ \omega = \frac{1-n}{n}, \tag{23}$$

$$u_4(x,y,t) = \left(\frac{b_0}{\exp[kx+y+\omega t]}\right)^{\frac{-1}{n-1}}, \quad k = \pm\sqrt{\frac{(n-1)^2}{an^2} - 1},\ \omega = n-1. \tag{24}$$

To compare our results with those obtained in [1], we set the parameters in Eq. (23), respectively, as follows:

$$(1) \quad a_0 = 1, \qquad k = \sqrt{\frac{(n-1)^2}{an^2} - 1}, \quad a = \frac{(n-1)^2}{2n^2};$$

$$(2) \quad a_0 = -1, \qquad k = \sqrt{\frac{(n-1)^2}{an^2} - 1}, \quad a = \frac{(n-1)^2}{2n^2};$$

$$(3) \quad a_0 = 1, \qquad k = -\sqrt{\frac{(n-1)^2}{an^2} - 1}, \quad a = \frac{(n-1)^2}{2n^2};$$

$$(4) \quad a_0 = -1 \qquad k = -\sqrt{\frac{(n-1)^2}{an^2} - 1}, \quad a = \frac{(n-1)^2}{2n^2}.$$

We therefore obtain

$$u_{3(1)}(x,y,t) = \left(1 + \exp\left[\frac{(n-1)}{n\sqrt{2a}}\left(x + y - \sqrt{2a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{n-1}{2n\sqrt{2a}}\left(x + y - \sqrt{2a}t\right)\right]\right)^{\frac{-1}{n-1}}$$

$$u_{3(2)}(x,y,t) = \left(1 - \exp\left[\frac{(n-1)}{n\sqrt{2a}}\left(x + y - \sqrt{2a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} - \frac{1}{2}\coth\left[\frac{n-1}{2n\sqrt{2a}}\left(x + y - \sqrt{2a}t\right)\right]\right)^{\frac{-1}{n-1}}$$

$$u_{3(3)}(x,y,t) = \left(1 + \exp\left[\frac{(1-n)}{n\sqrt{2a}}\left(x + y + \sqrt{2a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{n-1}{2n\sqrt{2a}}\left(x + y + \sqrt{2a}t\right)\right]\right)^{\frac{-1}{n-1}}$$

$$u_{3(4)}(x,y,t) = \left(1 - \exp\left[\frac{(1-n)}{n\sqrt{2a}}\left(x + y + \sqrt{2a}t\right)\right]\right)^{\frac{1}{n-1}} = \left(\frac{1}{2} + \frac{1}{2}\coth\left[\frac{n-1}{2n\sqrt{2a}}\left(x + y + \sqrt{2a}t\right)\right]\right)^{\frac{-1}{n-1}}.$$

These are the solutions obtained by the tanh method in [1].

4. Conclusion

The exp-function method was successfully used to obtain new generalized solitary solutions to the two generalized forms of the heat conduction equations. Our results include and make uniform all solutions of Wazwaz's in [1]. The solving procedure reveals that the exp-function method is a straightforward, concise and promising tool for solving nonlinear equations.

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